

# Noise-induced attractor annihilation in the delayed feedback logistic map

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## ABSTRACT

We study dynamics of the bistable logistic map with delayed feedback, under the influence of white Gaussian noise and periodic modulation applied to the variable. This system may serve as a model to describe population dynamics under finite resources in noisy environment with seasonal fluctuations. While a very small amount of noise has no effect on the global structure of the coexisting attractors in phase space, an intermediate noise totally eliminates one of the attractors. Slow periodic modulation enhances the attractor annihilation.

## 1. Introduction

Since noise is inevitably present in any natural system, the interaction between stochasticity and nonlinearity is extremely important in modeling dynamical behavior of different systems, including radiophysical [1], climatic [2–5], geophysical [6], epidemic [7], and optical models [8]. More subtly, the lack of detailed knowledge of biological/chemical/physical mechanisms underlying the dynamics, *a priori* stochastic perspective may allow us to study or expose many aspects of the system dynamical behavior. The system response to stochastic modulation depends on the noise character and amplitude as well as on the system nonlinear properties, its robustness and stability.

Several experimental and theoretical works have demonstrated different effects of noise on a multistable system, e.g., it induces stochastic [9], coherence [10] and vibrational [11] resonances, gives preference for some attractors [12–15], produces attractor hopping [16,17], and enhances multistability [18,19]. Pre-bifurcation noise amplification can serve as a precursor of forthcoming natural catastrophes [20,21]. Slow random fluctuations in a multistable system can be responsible for the appearance of rare large-amplitude pulses known as rogue waves [22].

Some researchers state that noise can change the system stability [23], either by increasing a disorder [24], including a transition to chaos [25–27], or, on the contrary, by inducing an order [28–32]. Since noise provokes various phase transitions [33,34], the system under stochastic modulation undergoes different bifurcations [13,

35,36], so that basins of attraction of all (or some) stable states (attractors) merge. The effect of noise on statistical properties of non-hyperbolic attractors of two coupled logistic maps has been studied by Anischenko et al. [34], who have shown that even a very small noise leads to the transition that manifests itself as attractor destruction. The probability properties of noisy attractors are also very sensitive to slightest changes in system parameters [37].

A multistable system displays quite distinct behavior for weak or strong noise. The difference in the effects of weak and strong noise on the attractor's structure has been well established experimentally in a multistable fiber laser [17,21]. Whereas large-amplitude noise merges the basins of attraction of coexisting states so that a trajectory visits intermittently different regimes resulting in a new intermittent attractor (so-called *attractor hopping* phenomenon), low-amplitude noise does not induce phase transitions and the system remains in one of the coexisting attractors [13]. Nevertheless, any noise, regardless of its amplitude, makes the system dynamics intrinsically probabilistic, an important requirement to allow a stochastic equations description. This means that even for very small noise, the attractors are only stable statistically, i.e., starting from the same initial conditions in the vicinity of a basin boundary, the system will with a certain probability undertake a trajectory to reach one or another coexisting attractor. These regions of the basins of attraction are associated with sets of given probabilities to belong to each of the coexisting attractors, meaning that with fixed initial conditions a different asymptotic state may be obtained for every calculation event.

In this work we study the influence of weak noise on the attractor structure in the bistable logistic map with delayed feedback. This research is a further extension of the application of the method for attractor annihilation by either periodic [38,39] or stochastic [40] forcing. Unlike other control methods (e.g., [41,42])

which result in stabilization of an unstable periodic orbit embedded into a chaotic attractor, our control eliminates one of the coexisting stable periodic orbits in a multistable system so that the system becomes monostable. Here, we focus on a particular system, the logistic map, which can be used as a model for population dynamics under finite resources in noisy environment with seasonal fluctuations. An important difference of the logistic map from previously studied systems is that the former is a one-dimensional system in which bistability is induced by a delayed feedback. We show how basins of attraction of coexisting attractors deform under random fluctuations and in combination with slow periodic perturbation.

## 2. Bistability in the logistic map

The logistic map is a prototype dynamical system, which for certain parameter values exhibits the coexistence of attractors. Dynamics of the logistic map has been of a great interest to biologists, following early work in ecology [43]. This map has long been used as a discrete-time model of population dynamics under finite resources. The model is given by the recursion

$$x_{n+1} = ax_n(1 - x_n), \quad (1)$$

where  $x_n \in [0, 1]$  and  $a \in [0, 4]$  are the system variable and parameter, respectively, and  $n$  is the iteration number or time. In the model of population dynamics,  $x$  and  $a$  denote, respectively, the population density and growth rate and  $n$  is referred to as the generation number; the population cannot exceed a certain size because of resource limitations. Given an initial state  $x_0$ , we obtain the state  $x_n$  after  $n$  iterations. Despite its inherent simplicity, the logistic map (1) has become an important corner stone of theoretical population biology and ecology [44]. It has also been applied to experimental and wild population (in particular, agricultural pests) [45]. More recently, this model has been used to describe biochemical systems dynamics in bioreactors [46–48]. The logistic map (1) exhibits apparently unpredictable behavior when the parameter (growth rate) exceeds 3.58, i.e., a strictly deterministic process for  $a \geq 3.58$  becomes chaotic.

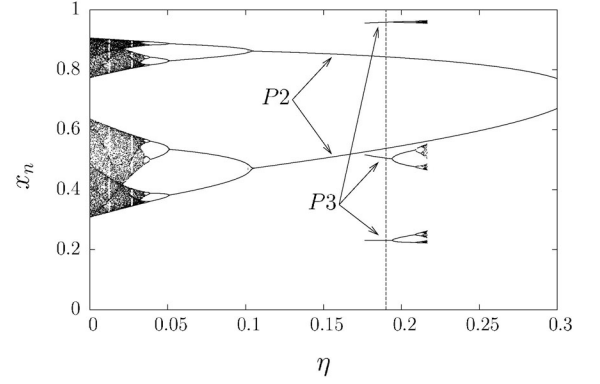
Although the simple logistic map (1) has only one stable solution, a short time-delayed feedback can induce bistability [49]. As known, time-delayed models describe many natural phenomena, in particular, in ecology. Hutt [50] was the first ecologist, who investigated the role of explicit delays in ecological models by considering the differential logistic equation

$$dx(t)/dt = x(t)(a - bx(t - \tau)), \quad (2)$$

with delay time  $\tau$ . Since the Hutt's work, delay differential equations have occupied attention of a great number of ecologists and mathematicians (for various models of such equations in ecology see, e.g., [51]). Here, in contrast to this logistic differential equation (2), we assume hypothetically that the species density at time  $n + 1$  depends on the difference between the density at the earlier generation by delay  $\tau$  and the previous generation, i.e., the dynamics is described as follows [49]:

$$x_{n+1} = ax_n(1 - x_n) + \eta(x_{n-\tau} - x_n), \quad (3)$$

where  $\eta$  is the delayed feedback strength, which can be either positive ( $\eta > 0$ ) or negative ( $\eta < 0$ ). A positive feedback is more appropriate for the population model because this suggests that previous populations (grandparents) are still reproducible. A form similar to Eq. (3) has been used by Pyragas [52], who stabilized unstable periodic orbits of period  $\tau$  and its subharmonics embedded into a chaotic attractor. Indeed, Schleya and Bees [53] showed that mechanisms associated with the slugs' time-delayed population



**Fig. 1.** Bifurcation diagram of delayed feedback logistic map (3) for  $a = 3.625$  and  $\tau = 1$ . For  $\eta = 0.19$  (vertical line) there coexist two stable solutions,  $P_2$  and  $P_3$ , which are found for different initial conditions.

dynamics can be responsible for large variations in numbers. They found that in all cases the delay term is of considerable qualitative importance in models which incorporate seasonal fluctuations. Moreover, they highlighted the fact that the delayed models are capable to produce a large range of solution behavior.

The feedback logistic map (3) with a short time delay displays bistability which is clearly seen in the bifurcation diagram in Fig. 1, where the feedback strength  $\eta$  is used as a control parameter. When the parameter  $a$  is fixed in a chaotic region ( $a = 3.625$ ), the coexistence of the period-2 ( $P_2$ ) and period-3 ( $P_3$ ) attractors for  $\tau = 1$  is detected within a certain range of  $\eta$ . In the following, we will study dynamics of the logistic map (3) for these fixed parameters:  $a = 3.625$ ,  $\eta = 0.19$ , and  $\tau = 1$ .

### 2.1. Effect of external noise on attractor annihilation

When external white Gaussian zero-mean environmental noise  $\xi_n$  of amplitude  $D$  is added at each iteration as follows

$$x_{n+1} = ax_n(1 - x_n) + \eta(x_{n-1} - x_n) + D\xi_n, \quad (4)$$

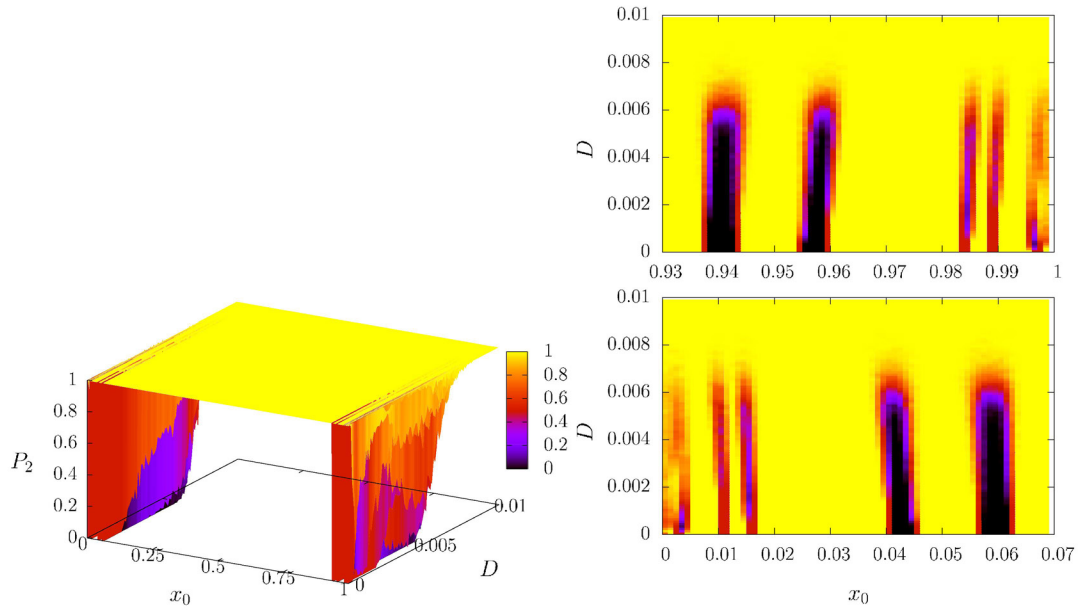
for biological applications, the generic model (4) allows, for example, a study of the life-time of molecular species involved in noisy feedback loops.

The probability to obtain one of the coexisting stable solutions ( $P_2$  or  $P_3$ ) depends on the noise amplitude  $D$ . Fig. 2 shows the basins of attraction of the coexisting  $P_2$  (yellow) and  $P_3$  (black) attractors for different  $D$ . Without noise ( $D = 0$ ), the map (4) is equal to the map (3) which exhibits the coexistence of the two periodic solutions. One can see that  $P_3$  is only found within the narrow windows of the initial conditions in the vicinity of  $x_0 = 0$  and  $x_0 = 1$ . The probabilities to find the period-2 ( $P_2$ ) and period-3 ( $P_3$ ) solutions complement each other, i.e., when  $P_2 = 1$  then  $P_3 = 0$ . The  $P_3$  solution is only statistically stable at  $D < 8 \times 10^{-3}$ . While  $D$  is increasing,  $P_3$  is decreasing and  $P_3$  becomes absolutely unstable at  $D > 8 \times 10^{-3}$  resulting in monostability.

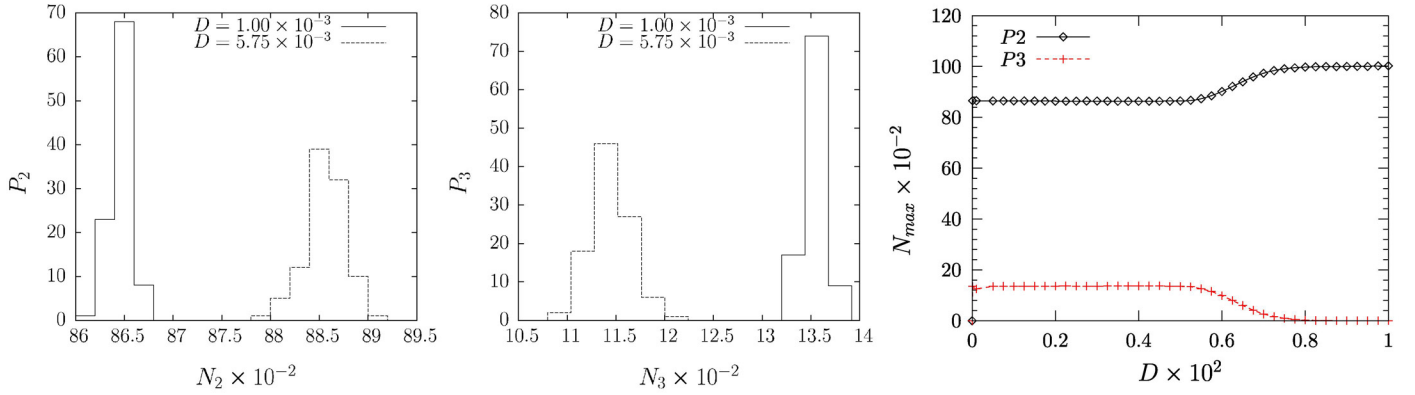
Since the basins' volumes (the number of initial conditions leading to the corresponding stable solution) have a probabilistic character, for every fixed noise amplitude  $D$  we calculate the basins' volumes 1000 times, every time exploring  $10^5$  initial conditions, and measure the probability distribution. The examples of the probability distributions  $P_2$  and  $P_3$  for the period-2 and period-3 attractors for two different noise amplitudes ( $D = 10^{-3}$  and  $D = 5.5 \times 10^{-3}$ ) are shown, respectively, in the left and middle panels of Fig. 3. One can see that for each noise amplitude, a certain basin's size appears with highest probability.

In the right panel of Fig. 3, we plot the total number of initial conditions (basin's volume) leading to the corresponding attractor with maximum probability,  $N_{\max}$ , as a function of  $D$ . One can see





**Fig. 2.** Probability properties of noisy delayed logistic map (4). (Left) Basins of attraction of period-2 (yellow) and period-3 (black) attractors and (right) their projections on  $(x_0, D)$  plane versus noise amplitude  $D$ . The color scale denotes the probability with which an initial condition leads to the period 2. The period-3 solution is absolutely unstable for  $D > 0.008$ . (For interpretation of the references to color in this figure, the reader is referred to the web version of this Letter.)



**Fig. 3.** Statistical characteristics of noisy delayed feedback logistic map (4). Probability distributions  $P_2$  (left) and  $P_3$  (middle) of P2 and P3 basins' sizes  $N_2$  and  $N_3$ , respectively, for two different noise amplitudes  $D$ . (Right) Most probable basins' sizes  $N_{\max}$  of P2 and P3 as functions of  $D$ .

that for small noise ( $D < 5 \times 10^{-3}$ ), the most probable basins' sizes are independent of noise, whereas for larger noise the P2 basin size enlarges as  $D$  is increased, while the P3 basin size shrinks to zero, giving rise to monostability for  $D > 0.008$ . Thus, we demonstrate that bistability in the logistic map can be easily eliminated by increasing the noise amplitude.

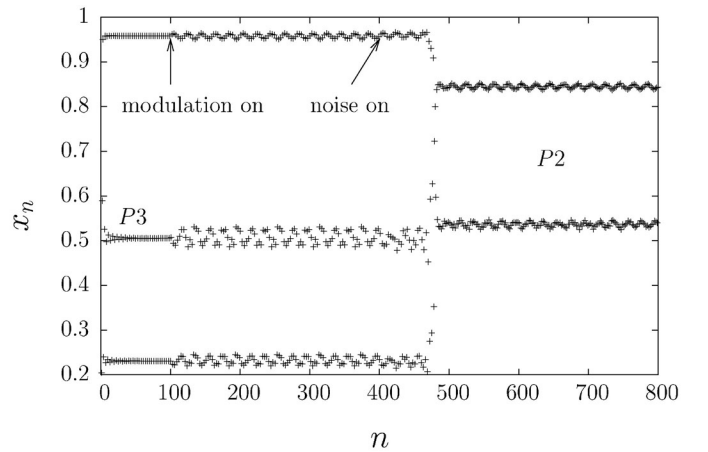
### 3. Effect of periodic modulation

In nature, the species population undergoes periodic seasonal changes of several environment parameters, such as, average daily temperature, daily time duration, moon phase, etc. To study the influence of the slow periodic modulation on the population dynamics, we add a harmonic term to our model (4) as

$$x_{n+1} = ax_n(1 - x_n) + \eta(x_{n-1} - x_n) + D\xi_n - \delta \sin(2\pi fn), \quad (5)$$

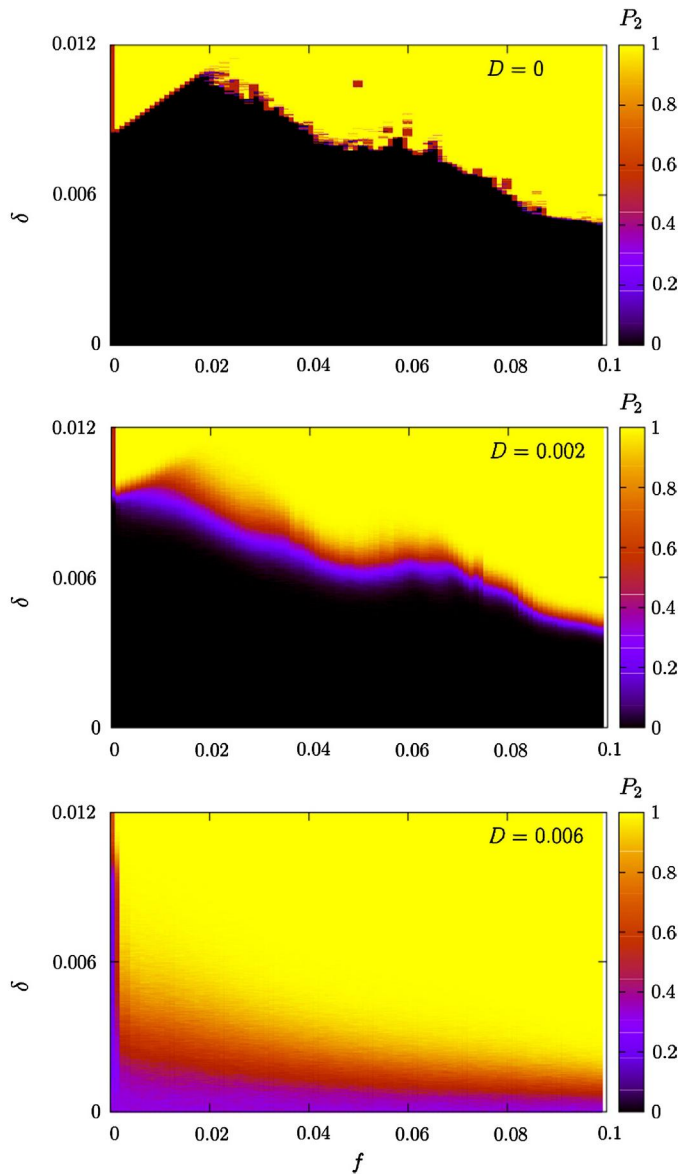
where  $\delta$  and  $f$  are, respectively, the modulation amplitude and frequency. Here, we suppose that  $\delta$  is so small that no qualitative changes occur in the stationary case when  $f \approx 0$ .

First, we iterate the system (3) (in the absence of noise and periodic modulation) from the initial condition  $x_0$  corresponding to the P3 attractor. After 100 iterations ( $n = 100$ ), we apply har-



**Fig. 4.** Time series demonstrated combined effect of noise and harmonic modulations in delayed feedback logistic map (5). Small harmonic modulation applied to the map being in the P3 attractor, after 100 iterations is not capable to destroy it, while in the presence of small noise added after 400 iterations, does this.

monic modulation to the variable, and then, after 400 iterations ( $n = 400$ ), we add noise, as shown in Fig. 4. The system, being



**Fig. 5.** State diagrams of noisy delayed feedback logistic map under harmonic modulation (5) in  $(f, \delta)$  parameter space for three different noise amplitudes: (top)  $D = 0$ , (middle)  $D = 0.002$ , and (bottom)  $D = 0.006$  for  $x_0 = 0.94$ ,  $\alpha = 3.625$ , and  $\eta = 0.19$ . The color shows the average probability over all initial conditions to target a trajectory to  $P_2$ . Noise facilitates annihilation of the  $P_3$  attractor in the presence of harmonic modulation. (For interpretation of the references to color in this figure, the reader is referred to the web version of this Letter.)

initially in  $P_3$ , after transients is either attracted or not attracted to  $P_2$ . The probability for being attracted to  $P_2$  depends on the noise amplitude  $D$ .

The effect of the periodic modulation is illustrated in Fig. 5 which shows the state diagrams in the  $(f, \delta)$  parameter space of the control modulation without noise [Fig. 5 (top)] and for two values of the noise amplitude:  $D = 2 \times 10^{-3}$  [Fig. 5 (middle)] and  $D = 6 \times 10^{-3}$  [Fig. 5 (bottom)]. For each value of the control parameters  $(f, \delta)$  we calculate the probability  $P_2$  of the system (5) to be attracted to  $P_2$  starting from initial conditions for  $P_3$ .  $P_2 = 1$  (yellow region) means that every trajectory is only attracted to  $P_2$ , i.e., for these modulation parameters the system is monostable. One can see that without noise ( $D = 0$ ) [Fig. 5 (top)], the border between the bistable (black) and monostable (yellow) regions is well defined (annihilation curve or crisis bifurcation line), whereas noise degrades the border and shifts it towards a lower values of  $\delta$ .

Thus, the combination of the random and periodic modulations produces more pronounced effect on the attractor annihilation to provide monostability more easily.

#### 4. Conclusions

The dynamics of the delayed feedback logistic map has been studied under the influence of external noise and periodic modulation applied to the system variable. Both the probability distribution and the most probable basins' size of the coexisting attractors depend on the noise amplitude and on the parameters of external modulation as well. The statistical analysis performed in this Letter is, we believe, the best way to exhibit how random perturbations affect the global structure of attractors in phase space. Given the importance the logistic map holds as a canonical model for dynamical systems, and in particular, as a model to describe population dynamics under finite resources in noisy environment, the present research will hopefully lead the way for better understanding the natural behavior of some species. On the other hand, similar approach can be applied to control multistability in other, more complex systems, e.g. multimodal maps [54].

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